

# SPECTRAL PROPERTIES OF STATIONARY SOLUTIONS OF THE NONLINEAR HEAT EQUATION

THIERRY CAZENAVE, FLÁVIO DICKSTEIN, AND FRED B. WEISSLER

*Abstract*

In this paper, we prove that if  $\Psi$  is a radially symmetric, sign-changing stationary solution of the nonlinear heat equation

$$(NLH) \quad u_t - \Delta u = |u|^\alpha u,$$

in the unit ball of  $\mathbb{R}^N$ ,  $N = 3$ , with Dirichlet boundary conditions, then the solution of (NLH) with initial value  $\lambda\Psi$  blows up in finite time if  $|\lambda - 1| > 0$  is sufficiently small and if  $\alpha > 0$  is sufficiently small. The proof depends on showing that the inner product of  $\Psi$  with the first eigenfunction of the linearized operator  $L = -\Delta - (\alpha + 1)|\Psi|^\alpha$  is nonzero.

## 1. Introduction

In this paper, we continue our study [1] [2], [3] of the instability of sign-changing stationary solutions of the nonlinear heat equation. Let  $\Omega$  be a bounded, smooth domain of  $\mathbb{R}^N$  and  $\alpha > 0$ . Given an initial value  $u_0 \in C_0(\Omega)$ , consider the nonlinear heat equation

$$(1.1) \quad \begin{cases} u_t - \Delta u = |u|^\alpha u, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0. \end{cases}$$

It is well known that the above initial value problem is locally well-posed. More precisely, there exists a maximal time  $0 < T_{u_0} \leq \infty$  and a (unique) function  $u \in C([0, T_{u_0}), C_0(\Omega)) \cap C((0, T_{u_0}), C^2(\overline{\Omega})) \cap C^1((0, T_{u_0}), C_0(\Omega))$  which is a classical solution of (1.1) on  $(0, T_{u_0})$  and such that  $u(0) = u_0$ . Furthermore, there is the blowup alternative: either  $T_{u_0} = \infty$  (i.e.  $u$  is a global solution) or else  $T_{u_0} < \infty$  and  $\lim_{t \uparrow T_{u_0}} \|u(t)\|_{L^\infty} = \infty$  (i.e.  $u$  blows up in finite time). An important question which has been studied

2000 *Mathematics Subject Classification.* 35K55, 35B35, 35J60.

*Key words.* Semilinear heat equation, finite-time blowup, sign-changing stationary solutions, linearized operator.

Flávio Dickstein was partially supported by CNPq (Brasil).

extensively over the past fifty years is to determine whether or not a solution blows up in finite time in terms of conditions on the initial value  $u_0$ . The case of positive solutions is better understood than the case of sign-changing solutions. Early papers showing the existence of blowing-up solutions are Kaplan [9], Itô [7], [8] and Fujita [4], [5]. To the best of our knowledge, the first blow-up result which applies to sign-changing solutions is due to Levine [10].

The question of finite-time blowup is related to the properties of stationary solutions. In particular, in the subcritical case

$$(1.2) \quad \alpha < \frac{4}{(N-2)^+},$$

it is well known that there exists a positive regular stationary solution  $\Psi$  of (1.1) and that if  $u_0 \geq \Psi$ ,  $u_0 \not\equiv \Psi$  then  $T_{u_0} < \infty$ . (See [11] and Section 19.2 in [12].) More recently, still in the subcritical case, Gazzola and Weth [6] proved that if  $u_0 \geq \Psi$ ,  $u_0 \not\equiv \Psi$  or if  $u_0 \leq \Psi$ ,  $u_0 \not\equiv \Psi$  where  $\Psi$  is a sign-changing stationary solution, then  $T_{u_0} < \infty$ .

The results in [1], [2], [3] show that there is a more subtle relationship between stationary solutions and blowup. More precisely, let  $\Psi \in C_0(\Omega)$  be a stationary solution of (1.1), i.e.

$$(1.3) \quad \begin{cases} -\Delta \Psi = |\Psi|^\alpha \Psi, \\ \Psi|_{\partial\Omega} = 0, \end{cases}$$

and consider the linearized operator  $L$  on  $L^2(\Omega)$  defined by

$$(1.4) \quad \begin{cases} D(L) = H^2(\Omega) \cap H_0^1(\Omega), \\ Lu = -\Delta u - (\alpha + 1)|\Psi|^\alpha u, \quad u \in D(L). \end{cases}$$

Let  $\lambda_1$  be the first eigenvalue of  $L$  and let  $\Phi$  be a corresponding positive eigenvector of  $L$ , i.e.

$$(1.5) \quad L\Phi = \lambda_1\Phi, \quad \Phi > 0.$$

The following theorem is a special case of Theorem 2.3 in [3].

**Theorem 1.1** ([3]). *Assume (1.2). Let  $\Psi$  and  $\Phi$  satisfy (1.3)–(1.5) and assume, in addition, that  $\Psi$  is sign-changing. If*

$$(1.6) \quad \int_{\Omega} \Psi \Phi \neq 0,$$

*then the solution of (1.1) with the initial value  $u_0 = \lambda\Psi$  blows up in finite time for all  $\lambda \neq 1$ ,  $\lambda$  sufficiently close to 1.*

Up until now, we have been able to establish (1.6) only for radially symmetric solutions in a ball. Recall that if  $\Psi$  is radially symmetric, then so is  $\Phi$ . In [1], we proved the following result which concerns values of  $\alpha$  close to  $4/(N-2)$  for  $N \geq 3$ .

**Theorem 1.2** ([1]). *Let  $\Omega$  be the unit ball of  $\mathbb{R}^N$  with  $N \geq 3$ . It follows that there exists  $0 < \bar{\alpha} < 4/(N-2)$  with the following property. If  $\bar{\alpha} < \alpha < 4/(N-2)$ ,  $\Psi$  and  $\Phi$  satisfy (1.3)–(1.5),  $\Psi$  is sign-changing and radially symmetric, then (1.6) holds. Therefore, there exists  $\varepsilon > 0$  such that if  $0 < |\lambda - 1| < \varepsilon$ , then the solution of (1.1) with the initial value  $u_0 = \lambda\Psi$  blows up in finite time.*

In this paper we prove the following result concerning small  $\alpha > 0$ .

**Theorem 1.3.** *Let  $\Omega$  be the unit ball of  $\mathbb{R}^N$  with  $N = 3$ . Given any positive integer  $\ell$ , there exists  $0 < \underline{\alpha} < 4/(N-2)$  with the following property. If  $0 < \alpha < \underline{\alpha}$ ,  $\Psi$  and  $\Phi$  satisfy (1.3)–(1.5),  $\Psi$  is radially symmetric and changes sign exactly  $\ell$  times on  $(0, 1)$  (as a function of  $r$ ), then (1.6) holds. Therefore, there exists  $\varepsilon > 0$  such that if  $0 < |\lambda - 1| < \varepsilon$ , then the solution of (1.1) with the initial value  $u_0 = \lambda\Psi$  blows up in finite time.*

In light of Theorem 1.1, Theorem 1.3 is an immediate consequence of the following two results. Both statements use the Cauchy principal value p.v. (see Section 4). The first one is valid in any dimension  $N \geq 1$ .

**Theorem 1.4.** *Let  $\Omega$  be the unit ball of  $\mathbb{R}^N$  and let  $\ell \geq 1$  be an integer. Assume that*

$$(1.7) \quad \text{p.v.} \int_0^1 \frac{\varphi}{\psi} |\psi'|^2 r^{N-1} dr \neq 0,$$

*for any two radially symmetric eigenfunctions  $\varphi, \psi$  of  $-\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$  such that  $\varphi > 0$  and  $\psi$  changes sign exactly  $\ell$  times on  $(0, 1)$  (as a function of  $r$ ). It follows that (1.6) holds for all sufficiently small  $\alpha > 0$  provided  $\Psi$  and  $\Phi$  satisfy (1.3)–(1.5) and  $\Psi$  is radially symmetric and changes sign exactly  $\ell$  times.*

**Proposition 1.5.** *Let  $\Omega$  be the unit ball of  $\mathbb{R}^N$  with  $N = 3$ . Let  $\varphi, \psi$  be two radially symmetric eigenfunctions of  $-\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$ . Suppose, in addition, that  $\varphi > 0$  and that  $\psi$  is sign-changing. It follows that (1.7) holds.*

**Remark 1.6.** In dimension  $N = 1$ , much more can be said. Indeed, given any  $\alpha > 0$ , every solution of (1.3) extends in an obvious way to an anti-periodic solution on  $\mathbb{R}$ . Consequently, the solution of (1.1) with

the initial value  $u_0 = \lambda\Psi$  is determined by the solution of (1.1), but on a subinterval where  $\Psi$  does not change sign (between two consecutive zeroes of  $\Psi$ ). In particular, the situation reduces to the case where  $\Psi > 0$ . Thus we see that, even if  $\Psi$  is sign-changing, the solution of (1.1) with the initial value  $u_0 = \lambda\Psi$  blows up if and only if  $|\lambda| > 1$ . Therefore, it follows from Theorem 1.1 that

$$(1.8) \quad \int_{\Omega} \Psi \Phi = 0,$$

for all  $\alpha > 0$  and for all  $\Psi$  and  $\Phi$  satisfying (1.3)–(1.5) with  $\Psi$  sign-changing. Since eigenfunctions of  $-\Delta$  in  $\Omega = (0, 1)$  are all multiples of  $\sin(k\pi r)$ ,  $k \geq 1$ , we deduce from (1.8) and Theorem 1.4 that

$$(1.9) \quad \text{p.v.} \int_0^1 \frac{\sin(\pi r)}{\sin((\ell+1)\pi r)} |\cos((\ell+1)\pi r)|^2 dr = 0,$$

for all  $\ell \geq 1$ . Note that if  $\ell$  is odd, then the integrand in (1.9) is anti-symmetric with respect to  $r = 1/2$ , from which (1.9) easily follows. On the other hand, we could not find any straightforward, direct proof of (1.9) for even integers  $\ell$ . Surprisingly, formula (1.9) plays an essential role in the proof of Proposition 1.5, which concerns the dimension  $N = 3$ .

The proof of Theorem 1.4 is based on the following idea. We consider the integral in (1.6) as a function of  $\alpha$ , where  $\Omega$  is a ball and  $\Psi$  is a radially symmetric solution of (1.3) which is positive at 0 and changes sign precisely  $\ell$  times. If we denote by  $g(\alpha)$  this integral, then it turns out that  $g(\alpha) \approx C\alpha^2$  as  $\alpha \downarrow 0$  where  $C$  is, up to a factor, given by the principal value integral (1.7). If  $N = 3$ , the radially symmetric eigenfunctions of  $-\Delta$  are given explicitly by  $r^{-1} \sin(\pi k r)$ , which enables us to prove Proposition 1.5.

The rest of the paper is organized as follows. In the following two sections, we prove respectively Theorem 1.4 and Proposition 1.5. The last section is an appendix where we recall the definition of principal value integrals and prove a convergence theorem which we use in the proof of Theorem 1.4.

## 2. Proof of Theorem 1.4

In order to prove Theorem 1.4, we reformulate the problem in terms of ordinary differential equations in the radial variable  $r$ . For convenience, we introduce a rescaling of the problem, which in fact depends on  $\alpha$ . More precisely, for any  $0 < \alpha < 4/(N-2)$ , we consider the solution  $\psi_\alpha$

of the equation

$$(2.1) \quad \begin{cases} \psi''_\alpha + \frac{N-1}{r} \psi'_\alpha + |\psi_\alpha|^\alpha \psi_\alpha = 0, \\ \psi_\alpha(0) = 1, \quad \psi'_\alpha(0) = 0. \end{cases}$$

Similarly, we consider the solution  $\psi_0$  of

$$(2.2) \quad \begin{cases} \psi''_0 + \frac{N-1}{r} \psi'_0 + \psi_0 = 0, \\ \psi_0(0) = 1, \quad \psi'_0(0) = 0. \end{cases}$$

It is well-known that for every  $0 \leq \alpha < 4/(N-2)$ ,  $\psi_\alpha$  oscillates indefinitely as  $r \rightarrow \infty$  and we denote by  $(\rho_{\alpha,j})_{j \geq 1}$  the increasing sequence of the zeros of  $\psi_\alpha$ . In other words,  $\psi_\alpha$  changes sign exactly  $j-1$  times in  $(0, \rho_{\alpha,j})$ . By local uniqueness of solutions of the ODE, it follows that

$$(2.3) \quad \psi'_\alpha(\rho_{\alpha,j}) \neq 0,$$

for all  $j \geq 1$  and all  $0 \leq \alpha < 4/(N-2)$ . We set

$$(2.4) \quad \Omega_{\alpha,j} = \{x \in \mathbb{R}^N; |x| < \rho_{\alpha,j}\}.$$

Given  $0 < \alpha < 4/(N-2)$ , we denote by  $L_{\alpha,j}$  the linearized operator  $-\Delta - (\alpha+1)|\psi_\alpha|^\alpha$  in  $L^2(\Omega_{\alpha,j})$  with Dirichlet boundary conditions. We let  $\lambda_{\alpha,j}$  be the first eigenvalue of  $L_{\alpha,j}$  and we call  $\varphi_{\alpha,j}$  the corresponding eigenvector normalized by the condition  $\varphi_{\alpha,j}(0) = 1$ . It follows that

$$(2.5) \quad \begin{cases} \varphi''_{\alpha,j} + \frac{N-1}{r} \varphi'_{\alpha,j} + (\alpha+1)|\psi_\alpha|^\alpha \varphi_{\alpha,j} + \lambda_{\alpha,j} \varphi_{\alpha,j} = 0, \\ \varphi_{\alpha,j}(0) = 1, \quad \varphi'_{\alpha,j}(0) = 0. \end{cases}$$

Similarly, we let  $L_{0,j} = -\Delta - I$  in  $L^2(\Omega_{0,j})$  with Dirichlet boundary conditions. We let  $\lambda_{0,j}$  be the first eigenvalue of  $L_{0,j}$  and we call  $\varphi_{0,j}$  the corresponding eigenvector normalized by the condition  $\varphi_{0,j}(0) = 1$ , so that

$$(2.6) \quad \begin{cases} \varphi''_{0,j} + \frac{N-1}{r} \varphi'_{0,j} + \varphi_{0,j} + \lambda_{0,j} \varphi_{0,j} = 0, \\ \varphi_{0,j}(0) = 1, \quad \varphi'_{0,j}(0) = 0. \end{cases}$$

Note that, given any  $0 \leq \alpha < 4/(N-2)$ ,  $\varphi_{\alpha,j}$  is defined originally on  $[0, \rho_{\alpha,j})$ , but as a solution of (2.5) (or (2.6) if  $\alpha = 0$ ) it can be extended to  $[0, \infty)$ .

Recall that the eigenvalues  $\lambda_{\alpha,j}$  are given by the variational principle

$$(2.7) \quad \begin{aligned} \lambda_{\alpha,j} &= \inf \{ u \in H_0^1(\Omega_{\alpha,j}), \|u\|_{L^2} = 1; (L_{\alpha,j}u, u)_{H^{-1}, H_0^1} \} \\ &= \inf \left\{ u \in H_0^1(\Omega_{\alpha,j}), \|u\|_{L^2} = 1; \int_{\Omega_{\alpha,j}} \{ |\nabla u|^2 - (\alpha+1)|\psi_\alpha|^\alpha u^2 \} \right\}. \end{aligned}$$

Note that  $(L_{\alpha,j}\psi_\alpha, \psi_\alpha)_{L^2} = -\alpha \int_{\Omega_{\alpha,j}} |\psi_\alpha|^{\alpha+2} < 0$  for all  $j \geq 1$  and  $0 < \alpha < 4/(N-2)$ , so that  $\lambda_{\alpha,j} < 0$ . Furthermore the function  $w(x) = \psi_0(\frac{\rho_{0,1}}{\rho_{0,j}}x)$  belongs to  $H_0^1(\Omega_{0,j})$ , is positive on  $\Omega_{0,j}$  and satisfies the equation  $-\Delta w - w = (\frac{\rho_{0,1}^2}{\rho_{0,j}^2} - 1)w$ , so that

$$(2.8) \quad \lambda_{0,j} = \frac{\rho_{0,1}^2}{\rho_{0,j}^2} - 1,$$

$$(2.9) \quad \varphi_{0,j}(x) \equiv \psi_0\left(\frac{\rho_{0,1}}{\rho_{0,j}}x\right),$$

for all  $j \geq 1$ .

**Lemma 2.1.**  $\psi_\alpha \rightarrow \psi_0$  in  $C^2([0, R])$  for all  $R > 0$  as  $\alpha \downarrow 0$ . Moreover,

$$(2.10) \quad \rho_{\alpha,j} \xrightarrow{\alpha \downarrow 0} \rho_{0,j},$$

for all  $j \geq 0$ .

*Proof:* Note that  $|x|^\alpha x \rightarrow x$  as  $\alpha \downarrow 0$ , uniformly for  $x$  in a bounded subset of  $\mathbb{R}$ . It easily follows that  $\psi_\alpha \rightarrow \psi_0$  in  $C^2([0, R])$  for all  $R > 0$ . The estimate (2.10) then follows from the convergence of  $\psi_\alpha$  to  $\psi_0$  in  $C^1$  and the nondegeneracy property (2.3).  $\square$

**Lemma 2.2.**  $\lambda_{\alpha,j} \rightarrow \lambda_{0,j}$  as  $\alpha \downarrow 0$  for all  $j \geq 1$ .

*Proof:* Let  $\mu_{\alpha,j}$  be the first eigenvalue of  $-\Delta - I$  in  $L^2(\Omega_{\alpha,j})$  with Dirichlet boundary conditions, i.e.

$$(2.11) \quad \mu_{\alpha,j} = \inf \left\{ u \in H_0^1(\Omega_{\alpha,j}), \|u\|_{L^2} = 1; \int_{\Omega_{\alpha,j}} \{ |\nabla u|^2 - u^2 \} \right\}.$$

It follows from (2.10) and a straightforward rescaling argument that

$$(2.12) \quad \mu_{\alpha,j} \xrightarrow{\alpha \downarrow 0} \lambda_{0,j}.$$

It thus suffices to show that  $\lambda_{\alpha,j} - \mu_{\alpha,j} \rightarrow 0$  as  $\alpha \downarrow 0$ . Since  $\psi_\alpha \rightarrow \psi_0$  in  $C^1([0, R])$  for all  $R > 0$  by Lemma 2.1, we see that  $(\alpha+1)|\psi_\alpha|^\alpha \rightarrow 1$  as  $\alpha \downarrow 0$  in  $L^p(0, R)$  for every  $R > 0$  and  $p < \infty$ . It then follows from

standard variational arguments that the infimum in (2.7) converges to the infimum in (2.11), and this completes the proof.  $\square$

**Lemma 2.3.**  $\varphi_{\alpha,j} \rightarrow \varphi_{0,j}$  in  $C^1([0, R])$  as  $\alpha \downarrow 0$  for all  $R > 0$ .

*Proof:* We deduce from equations (2.5) and (2.6) that

$$(2.13) \quad \begin{aligned} \varphi_{\alpha,j}(r) - \varphi_{0,j}(r) = & - \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} [(\alpha+1)|\psi_\alpha|^\alpha + \lambda_{\alpha,j}] (\varphi_{\alpha,j} - \varphi_{0,j}) \\ & - \int_0^r \frac{1}{s^{N-1}} \int_0^s \sigma^{N-1} [(\alpha+1)|\psi_\alpha|^\alpha - 1 + \lambda_{\alpha,j} - \lambda_{0,j}] \varphi_{0,j}, \end{aligned}$$

for all  $r > 0$ . Since the last integral in (2.13) converges to 0 as  $\alpha \downarrow 0$  by Lemmas 2.1 and 2.2, it easily follows from Gronwall's inequality that  $\varphi_{\alpha,j} \rightarrow \varphi_{0,j}$  in  $C([0, R])$  as  $\alpha \downarrow 0$  for all  $R > 0$ . Next, we deduce from the equations (2.5) and (2.6) that

$$(2.14) \quad \begin{aligned} \varphi'_{\alpha,j}(r) - \varphi'_{0,j}(r) \\ = - \frac{1}{r^{N-1}} \int_0^r s^{N-1} \{ [(\alpha+1)|\psi_\alpha|^\alpha + \lambda_{\alpha,j}] \varphi_{\alpha,j} - [1 + \lambda_{0,j}] \varphi_{0,j} \}, \end{aligned}$$

for all  $r > 0$ . Applying Lemmas 2.1 and 2.2, and the uniform convergence on bounded sets of  $\varphi_{\alpha,j}$  to  $\varphi_{0,j}$ , we deduce that  $\varphi'_{\alpha,j} \rightarrow \varphi'_{0,j}$  in  $C([0, R])$  as  $\alpha \downarrow 0$  for all  $R > 0$ . This completes the proof.  $\square$

We now set

$$(2.15) \quad \mathcal{J}_j(\alpha) = \int_{\Omega_{\alpha,j}} \psi_\alpha \varphi_{\alpha,j},$$

for all  $j \geq 1$  and  $0 \leq \alpha < 4/(N-2)$ . Note that, multiplying the equation for  $\psi_\alpha$  by  $\varphi_{\alpha,j}$  and the equation for  $\varphi_{\alpha,j}$  by  $\psi_\alpha$ ,

$$(2.16) \quad \frac{-\lambda_{\alpha,j}}{\alpha} \mathcal{J}_j(\alpha) = \int_{\Omega_{\alpha,j}} |\psi_\alpha|^\alpha \psi_\alpha \varphi_{\alpha,j}.$$

Next, it follows from (2.5) that

$$(2.17) \quad -\lambda_{\alpha,j} \varphi_{\alpha,j} = \Delta \varphi_{\alpha,j} + (\alpha+1) |\psi_\alpha|^\alpha \varphi_{\alpha,j},$$

so that by (2.16)

$$\begin{aligned}
 \frac{\lambda_{\alpha,j}^2}{\alpha} \mathcal{J}_j(\alpha) &= \int_{\Omega_{\alpha,j}} |\psi_\alpha|^\alpha \psi_\alpha (\Delta \varphi_{\alpha,j} + (\alpha+1) |\psi_\alpha|^\alpha \varphi_{\alpha,j}) \\
 (2.18) \quad &= \int_{\Omega_{\alpha,j}} [\Delta(|\psi_\alpha|^\alpha \psi_\alpha) + (\alpha+1) |\psi_\alpha|^{2\alpha} \psi_\alpha] \varphi_{\alpha,j} \\
 &= \alpha(\alpha+1) \int_{\Omega_{\alpha,j}} |\psi_\alpha|^{\alpha-2} \psi_\alpha \varphi_{\alpha,j} |\nabla \psi_\alpha|^2,
 \end{aligned}$$

where we used the equation for  $\psi_\alpha$ . Note that the last integral makes sense because  $\psi_\alpha$  is a radial solution, so when  $\psi_\alpha$  vanishes  $\psi'_\alpha$  does not, thus  $|\psi_\alpha|^{\alpha-2} \psi_\alpha$  is integrable. Therefore,

$$\begin{aligned}
 \frac{1}{\alpha+1} \frac{\lambda_{\alpha,j}^2}{\alpha^2} \mathcal{J}_j(\alpha) &= \int_{\Omega_{\alpha,j}} |\psi_\alpha|^{\alpha-2} \psi_\alpha \varphi_{\alpha,j} |\nabla \psi_\alpha|^2 \\
 (2.19) \quad &= \int_{\Omega_{\alpha,j}} |\psi_\alpha|^\alpha \frac{\varphi_{\alpha,j}}{\psi_\alpha} |\nabla \psi_\alpha|^2.
 \end{aligned}$$

Since the functions are radially symmetric, we obtain

$$(2.20) \quad \frac{1}{c_N} \frac{1}{\alpha+1} \frac{\lambda_{\alpha,j}^2}{\alpha^2} \mathcal{J}_j(\alpha) = \int_0^{\rho_{\alpha,j}} |\psi_\alpha|^\alpha \frac{\varphi_{\alpha,j}}{\psi_\alpha} |\psi'_\alpha|^2 r^{N-1} dr,$$

where  $c_N$  is the  $(N-1)$ -dimensional measure of the unit sphere in  $\mathbb{R}^N$ . We claim that

$$(2.21) \quad \int_0^{\rho_{\alpha,j}} |\psi_\alpha|^\alpha \frac{\varphi_{\alpha,j}}{\psi_\alpha} |\psi'_\alpha|^2 r^{N-1} dr \xrightarrow[\alpha \downarrow 0]{\text{p.v.}} \int_0^{\rho_{0,j}} \frac{\varphi_{0,j}}{\psi_0} |\psi'_0|^2 r^{N-1} dr,$$

where p.v. denotes the principal value. This can be proved using Lemma 4.1 with  $f_\alpha(r) = \varphi_{\alpha,j} |\psi'_\alpha|^2 r^{N-1}$  and  $g_\alpha(r) = \psi_\alpha$ . The principal value integral in (2.21) needs to be expressed as the sum of integrals on smaller intervals, where each interval contains in its interior one zero of  $\psi_0$ .

We now complete the proof of Theorem 1.4, and so we let  $\Omega$  be the unit ball of  $\mathbb{R}^N$ , we fix an integer  $\ell \geq 1$  and we set

$$j = \ell + 1.$$

Let  $\varphi, \psi$  be two radially symmetric eigenfunctions of  $-\Delta$  with domain  $H^2(\Omega) \cap H_0^1(\Omega)$  and suppose that  $\varphi > 0$  and  $\psi$  changes sign exactly  $\ell$  times on  $(0, 1)$  (as a function of  $r$ ). It follows that there exist two constants  $a, b \neq 0$  such that

$$\varphi(r) \equiv a \varphi_{0,j}(\rho_{0,j} r), \quad \psi(r) \equiv b \psi_0(\rho_{0,j} r).$$



In particular (1.7) is equivalent to

$$(2.22) \quad \text{p.v.} \int_0^{\rho_{0,j}} \frac{\varphi_{0,j}}{\psi_0} |\psi_0'|^2 r^{N-1} dr \neq 0.$$

We deduce from (2.20), (2.21) and (2.22) that

$$(2.23) \quad \mathcal{J}_j(\alpha) \neq 0,$$

for all sufficiently small  $\alpha > 0$ . Finally, let  $\Psi$  and  $\Phi$  satisfy (1.3)–(1.5) and suppose  $\Psi$  is radially symmetric and changes sign exactly  $\ell$  times. It follows that there exist constants  $c, d \neq 0$  such that

$$\Phi(r) \equiv c\varphi_{\alpha,j}(\rho_{\alpha,j}r), \quad \Psi(r) \equiv d\psi_\alpha(\rho_{\alpha,j}r).$$

Therefore, (1.6) is equivalent to (2.23). This proves Theorem 1.4.

### 3. Proof of Proposition 1.5

We continue with the notation introduced in Section 2 and we set

$$(3.1) \quad \mathcal{U} = \text{p.v.} \int_0^{\rho_{0,j}} \frac{\varphi_{0,j}}{\psi_0} |\psi_0'|^2 r^{N-1} dr.$$

Note that in dimension  $N = 3$ ,

$$\psi_0(r) \equiv \frac{\sin r}{r},$$

$$\rho_{0,j} = j\pi,$$

$$\lambda_{0,j} = -\frac{j^2 - 1}{j^2},$$

$$\varphi_{0,j}(r) \equiv j \frac{\sin(r/j)}{r},$$

so that

$$(3.2) \quad \mathcal{U} = j \text{ p.v.} \int_0^{j\pi} \frac{\sin(r/j)}{\sin r} (r \cos r - \sin r)^2 \frac{dr}{r^2}.$$

Expanding the square, we deduce that

$$(3.3) \quad \begin{aligned} \mathcal{U} = j \text{ p.v.} \int_0^{j\pi} \frac{\sin(r/j)}{\sin r} \cos^2 r dr \\ - 2j \int_0^{j\pi} \sin(r/j) \cos r \frac{dr}{r} + j \int_0^{j\pi} \sin(r/j) \sin r \frac{dr}{r^2}. \end{aligned}$$

(We note that the last two integrals are ordinary integrals because there is no singularity.) Since

$$(3.4) \quad \text{p.v.} \int_0^{j\pi} \frac{\sin(r/j)}{\sin r} \cos^2 r \, dr = 0,$$

for all  $j \geq 2$  by (1.9), it follows that

$$(3.5) \quad \begin{aligned} \mathcal{U} &= -2j \int_0^{j\pi} \sin(r/j) \cos r \frac{dr}{r} + j \int_0^{j\pi} \sin(r/j) \sin r \frac{dr}{r^2} \\ &= -j \int_0^{j\pi} \sin(r/j) \cos r \frac{dr}{r} - j \int_0^{j\pi} \sin(r/j) (r \cos r - \sin r) \frac{dr}{r^2}. \end{aligned}$$

Furthermore, since

$$\left( \frac{\sin r}{r} \right)' = \frac{1}{r^2} (r \cos r - \sin r),$$

we deduce that

$$(3.6) \quad \begin{aligned} \mathcal{U} &= -j \int_0^{j\pi} \sin(r/j) \cos r \frac{dr}{r} + \int_0^{j\pi} \cos(r/j) \sin r \frac{dr}{r} \\ &= \frac{j^2-1}{2} \left[ \frac{1}{j-1} \int_0^{j\pi} \sin((j-1)r/j) \frac{dr}{r} - \frac{1}{j+1} \int_0^{j\pi} \sin((j+1)r/j) \frac{dr}{r} \right] \\ &= \frac{j^2-1}{2} \left[ \frac{1}{j-1} \int_0^{(j-1)\pi} \sin r \frac{dr}{r} - \frac{1}{j+1} \int_0^{(j+1)\pi} \sin r \frac{dr}{r} \right] \\ &= \int_0^{(j-1)\pi} \sin r \frac{dr}{r} - \frac{j-1}{2} \int_{(j-1)\pi}^{(j+1)\pi} \sin r \frac{dr}{r}. \end{aligned}$$

Next, since the function  $r \mapsto 1/r$  is decreasing, it follows from easy calculations that

$$(3.7) \quad \int_{\ell\pi}^x \sin r \frac{dr}{r} > 0,$$

for all even integers  $\ell$  and all  $x > \ell\pi$  and that

$$(3.8) \quad \int_{\ell\pi}^x \sin r \frac{dr}{r} < 0,$$

for all odd integers  $\ell$  and all  $x > \ell\pi$ . If  $j$  is even, then applying (3.7) with  $\ell = 0$  and  $x = (j-1)\pi$  and (3.8) with  $\ell = j-1$  and  $x = (j+1)\pi$ , we see that  $\mathcal{U}$  is the sum of two positive terms, thus  $\mathcal{U} > 0$ . If  $j$  is odd

(and so  $j \geq 3$ ), then we apply (3.7) with  $\ell = 2$  and  $x = (j-1)\pi$  and we write

$$\begin{aligned}
 \mathcal{U} &\geq \int_0^{2\pi} \sin r \frac{dr}{r} - \frac{j-1}{2} \int_{(j-1)\pi}^{(j+1)\pi} \sin r \frac{dr}{r} \\
 &= \int_0^{2\pi} \sin r \frac{dr}{r} - \frac{j-1}{2} \int_0^{2\pi} \sin((j-1)\pi + r) \frac{dr}{(j-1)\pi + r} \\
 (3.9) \quad &= \int_0^{2\pi} \sin r \frac{dr}{r} - \frac{j-1}{2} \int_0^{2\pi} \sin r \frac{dr}{(j-1)\pi + r} \\
 &= \int_0^{2\pi} \sin r \left[ \frac{2r + (j-1)(2\pi - r)}{2r(r + (j-1)\pi)} \right] dr.
 \end{aligned}$$

We set

$$g(r) = \frac{2r + (j-1)(2\pi - r)}{2r(r + (j-1)\pi)},$$

so that

$$g'(r) = -\frac{(j-3)[2r(r + (j-1)\pi)] + [2r + (j-1)(2\pi - r)](4r + 2(j-1)\pi)}{4r^2(r + (j-1)\pi)^2}.$$

Thus we see that  $g'(r) < 0$  for all  $0 < r < 2\pi$ . Since  $g(2\pi) > 0$ , it follows that  $g$  is positive and decreasing on  $(0, 2\pi)$ . It follows easily that

$$\int_0^{2\pi} \sin r g(r) dr > 0.$$

Applying (3.9), we conclude that  $\mathcal{U} > 0$ , which proves the proposition.

#### 4. Appendix: principal value integrals

Let  $I$  be a closed interval of  $\mathbb{R}$  with 0 in its interior. We recall that if  $f \in C^1(I)$ ,  $g \in C^2(I)$ ,  $g(0) = 0$ ,  $g'(0) \neq 0$  and  $g(x) \neq 0$  for  $x \neq 0$ , then the Cauchy principal value integral

$$(4.1) \quad \text{p.v.} \int_I \frac{f(x)}{g(x)} dx = \lim_{\varepsilon \downarrow 0} \int_{I \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{g(x)} dx,$$

is well defined. Moreover if  $I$  is symmetric around 0, i.e.  $I = [-a, a]$  with  $a > 0$ , then

$$(4.2) \quad \text{p.v.} \int_{-a}^a \frac{f(x)}{g(x)} dx = \int_{-a}^a \theta(x) dx,$$

where

$$(4.3) \quad \theta(x) = \begin{cases} \frac{1}{x} \int_0^x h'(t) dt & x \neq 0, \\ h'(0) & x = 0, \end{cases}$$

and

$$(4.4) \quad h(x) = \frac{xf(x)}{g(x)},$$

for  $x \neq 0$ . To see this, note that the assumptions on  $f$  and  $g$  ensure that both  $h(x)$  and  $h'(x)$  have a limit as  $x \rightarrow 0$ , so that  $h$  extends to a  $C^1$  function on  $I$ . Thus we see that  $\theta$  is continuous and formula (4.2) follows by writing

$$\frac{f(x)}{g(x)} = \frac{h(0)}{x} + \theta(x).$$

The following lemma is crucial to our analysis. It may well follow from known facts about principal value integrals, but we could not find an appropriate reference. While the statement is natural, our proof is unfortunately somewhat technical. To see the essential ideas, we suggest that the reader set  $g_\alpha(x) = x$  in both the statement and the proof. The key observation in the proof is the trivial identity (4.22). Everything else is just technical embellishment.

**Lemma 4.1.** *Let  $I$  be a closed interval of  $\mathbb{R}$  with  $0 \in \text{int } I$ . For every  $\alpha \in [0, 1)$ , let  $x_\alpha \in \text{int } I$ ,  $f_\alpha \in C^1(I)$ ,  $g_\alpha \in C^2(I)$ ,  $g_\alpha(x_\alpha) = 0$ ,  $g'_\alpha(x_\alpha) \neq 0$  and  $g_\alpha(x) \neq 0$  for  $x \neq x_\alpha$ . Assume  $f_\alpha \rightarrow f_0$  in  $C^1(I)$ ,  $g_\alpha \rightarrow g_0$  in  $C^2(I)$  and  $x_\alpha \rightarrow 0$  as  $\alpha \downarrow 0$ . It follows that*

$$(4.5) \quad \int_I |g_\alpha(x)|^\alpha \frac{f_\alpha(x)}{g_\alpha(x)} dx \longrightarrow \text{p.v.} \int_I \frac{f_0(x)}{g_0(x)} dx,$$

as  $\alpha \downarrow 0$ .

*Proof:* We first assume that  $x_\alpha = 0$  for all  $\alpha \in [0, 1)$ . Fix  $a > 0$  such that  $[-a, a] \subset I$ . For  $\alpha \geq 0$  we set

$$(4.6) \quad \tilde{h}_\alpha(x) = \frac{xf_\alpha(x)}{g_\alpha(x)},$$

and

$$(4.7) \quad h_\alpha(x) = \tilde{h}_\alpha(x)|g_\alpha(x)|^\alpha,$$

for  $x \neq 0$ . (Note that  $\tilde{h}_0(x) = h_0(x)$  if  $x \neq 0$ .) The hypotheses on  $f_\alpha$  and  $g_\alpha$  imply that both  $\tilde{h}_\alpha$  and  $\tilde{h}'_\alpha$  have a limit as  $x \rightarrow 0$ , so that  $\tilde{h}_\alpha$  extends

to a  $C^1$  function on  $I$ . In addition,

$$(4.8) \quad h'_\alpha(x) = \tilde{h}'_\alpha(x)|g_\alpha(x)|^\alpha + \alpha\tilde{h}_\alpha(x)|g_\alpha(x)|^{\alpha-2}g'_\alpha(x)g'_\alpha(x),$$

for  $x \neq 0$ . It follows from the hypotheses on  $f_\alpha$  and  $g_\alpha$  and formulas (4.6), (4.7) and (4.8) that there exists a constant  $C$  such that

$$(4.9) \quad |\tilde{h}_\alpha(x)| + |x|^{-\alpha}|h_\alpha(x)| + |\tilde{h}'_\alpha(x)| + |x|^{1-\alpha}|h'_\alpha(x)| \leq C,$$

for all  $0 \leq \alpha \leq 1$  and  $x \in I$  and that

$$(4.10) \quad \tilde{h}_\alpha \xrightarrow[\alpha \downarrow 0]{} \tilde{h}_0 \text{ in } C^1(I).$$

In particular,  $h'_\alpha$  is in  $L^1(I)$  (in  $C^1(I)$  if  $\alpha = 0$ ) and we set

$$(4.11) \quad \theta_\alpha(x) = \frac{1}{x} \int_0^x h'_\alpha(t) dt,$$

for  $x \neq 0$  and  $\alpha \geq 0$ . It follows from (4.7) that if  $\alpha > 0$ , then  $h_\alpha$  extends to a Hölder continuous function on  $I$  with  $h_\alpha(0) = 0$  and that

$$(4.12) \quad h_\alpha(x) = \int_0^x h'_\alpha(t) dt.$$

We deduce from (4.7), (4.12) and (4.11) that if  $\alpha > 0$  then

$$(4.13) \quad \begin{aligned} \int_{-a}^a |g_\alpha(x)|^\alpha \frac{f_\alpha(x)}{g_\alpha(x)} dx &= \int_{-a}^a \frac{1}{x} h_\alpha(x) dx = \int_{-a}^a \frac{1}{x} \int_0^x h'_\alpha(t) dt dx \\ &= \int_{-a}^a \theta_\alpha(x) dx. \end{aligned}$$

It follows from (4.11) that

$$(4.14) \quad \int_{-a}^a [\theta_\alpha(x) - \theta_0(x)] dx = \int_{-a}^a \frac{1}{x} \int_0^x [h'_\alpha(t) - h'_0(t)] dt dx,$$

and using (4.8), we write for  $\alpha > 0$

$$(4.15) \quad \begin{aligned} \int_{-a}^a [\theta_\alpha(x) - \theta_0(x)] dx &= \int_{-a}^a \frac{1}{x} \int_0^x [\tilde{h}'_\alpha(t)|g_\alpha(t)|^\alpha - h'_0(t)] dt dx \\ &+ \int_{-a}^a \frac{1}{x} \int_0^x \alpha \tilde{h}_\alpha(t)|g_\alpha(t)|^{\alpha-2} g'_\alpha(t) g'_\alpha(t) dt dx =: A_1(\alpha) + A_2(\alpha). \end{aligned}$$

It is clear that the operator  $w \mapsto x^{-1} \int_0^x w(t) dt$  is continuous from  $L^2(-a, a)$  onto  $L^1(-a, a)$ . Moreover, since  $h_0 = \tilde{h}_0$ , it follows from (4.10) that  $\tilde{h}'_\alpha \rightarrow \tilde{h}'_0$  in  $C([-a, a])$ , so that by dominated convergence,

$$(4.16) \quad A_1(\alpha) \xrightarrow[\alpha \downarrow 0]{} 0.$$

Next, we write

$$(4.17) \quad \tilde{h}_\alpha(t) = \tilde{h}_\alpha(0) + \int_0^t \tilde{h}'_\alpha(s) ds,$$

so that

$$(4.18) \quad A_2(\alpha) = \int_{-a}^a \frac{1}{x} \int_0^x \alpha \left( \int_0^t \tilde{h}'_\alpha(s) ds \right) |g_\alpha(t)|^{\alpha-2} g_\alpha(t) g'_\alpha(t) dt dx \\ + \int_{-a}^a \frac{1}{x} \int_0^x \alpha \tilde{h}_\alpha(0) |g_\alpha(t)|^{\alpha-2} g_\alpha(t) g'_\alpha(t) dt dx =: B(\alpha) + C(\alpha).$$

It follows from the hypotheses on  $f_\alpha$  and  $g_\alpha$  that  $||g_\alpha(t)|^{\alpha-2} g_\alpha(t) g'_\alpha(t)| \leq C|t|^{\alpha-1}$  with  $C$  independent of  $\alpha$ . Applying (4.10), we deduce that

$$(4.19) \quad \left| \left( \int_0^t \tilde{h}'_\alpha(s) ds \right) |g_\alpha(t)|^{\alpha-2} g_\alpha(t) g'_\alpha(t) \right| \leq C|t|^\alpha,$$

with  $C$  independent of  $\alpha$ . Thus

$$(4.20) \quad |B(\alpha)| \leq C\alpha \int_{-a}^a |x|^\alpha \xrightarrow{\alpha \downarrow 0} 0.$$

Next, we have

$$(4.21) \quad C(\alpha) = \int_{-a}^a \frac{1}{x} \int_0^x \tilde{h}_\alpha(0) (|g_\alpha(t)|^\alpha)' dt dx = \tilde{h}_\alpha(0) \int_{-a}^a \frac{1}{x} |g_\alpha(x)|^\alpha dx \\ = \tilde{h}_\alpha(0) \left[ \int_{-a}^a \frac{1}{x} [|g_\alpha(x)|^\alpha - |g'_\alpha(0)|^\alpha |x|^\alpha] dx + \int_{-a}^a \frac{1}{x} |g'_\alpha(0)|^\alpha |x|^\alpha dx \right].$$

The function  $|x|^\alpha/x$  being in  $L^1(-a, a)$  and odd, we see that

$$(4.22) \quad \int_{-a}^a \frac{1}{x} |g'_\alpha(0)|^\alpha |x|^\alpha dx = 0.$$

It follows from (4.21)–(4.22) that

$$C(\alpha) = \tilde{h}_\alpha(0) \int_{-a}^a \frac{1}{x} [|g_\alpha(x)|^\alpha - |g'_\alpha(0)|^\alpha |x|^\alpha] dx.$$

Since  $||g_\alpha(x)| - |g'_\alpha(0)||x| \leq C|x|^2$  and  $\min\{|g_\alpha(x)|, |g'_\alpha(0)||x|\} \geq \eta|x|$  with  $C$  and  $\eta$  independent of  $\alpha$ , we deduce that

$$||g_\alpha(x)|^\alpha - |g'_\alpha(0)|^\alpha |x|^\alpha| \leq C\alpha|x|^{1+\alpha},$$

with  $C$  independent of  $\alpha$ . Therefore,

$$(4.23) \quad |C(\alpha)| \leq C\alpha \int_{-a}^a \frac{1}{|x|} |x|^{1+\alpha} \xrightarrow{\alpha \downarrow 0} 0.$$

It follows from (4.15), (4.16), (4.18), (4.20) and (4.23) that

$$(4.24) \quad \int_{-a}^a \theta_\alpha(x) dx \xrightarrow{\alpha \downarrow 0} \int_{-a}^a \theta_0(x) dx.$$

Applying formulas (4.13) and (4.2), we conclude that

$$(4.25) \quad \int_{-a}^a |g_\alpha(x)|^\alpha \frac{f_\alpha(x)}{g_\alpha(x)} dx \xrightarrow{\alpha \downarrow 0} \text{p.v.} \int_{-a}^a \frac{f_0(x)}{g_0(x)} dx.$$

Formula (4.5) follows, since clearly

$$(4.26) \quad \int_{I \setminus [-a, a]} |g_\alpha(x)|^\alpha \frac{f_\alpha(x)}{g_\alpha(x)} dx \xrightarrow{\alpha \downarrow 0} \int_{I \setminus [-a, a]} \frac{f_0(x)}{g_0(x)} dx.$$

This completes the proof in the case  $x_\alpha = 0$  for all  $\alpha \in [0, 1]$ . The general case follows easily by translating the integrals so that  $x_\alpha$  moves to 0. The error produced at the endpoints clearly converges to 0.  $\square$

**Acknowledgements.** The authors thank the referee for a very careful reading of the manuscript and constructive remarks.

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Thierry Cazenave:

Université Pierre et Marie Curie & CNRS  
 Laboratoire Jacques-Louis Lions  
 4 place Jussieu  
 75252 Paris Cedex 05  
 France  
*E-mail address:* `thierry.cazenave@upmc.fr`

Flávio Dickstein:

Instituto de Matemática  
 Universidade Federal do Rio de Janeiro  
 Caixa Postal 68530  
 21944-970 Rio de Janeiro, R.J.  
 Brazil  
*E-mail address:* `flavio@labma.ufrj.br`

Fred B. Weissler:

Université Paris 13  
 LAGA UMR CNRS 7539  
 99 Avenue J.-B. Clément  
 93430 Villetaneuse  
 France  
*E-mail address:* `weissler@math.univ-paris13.fr`

Primera versió rebuda el 23 de març de 2010,  
 darrera versió rebuda el 13 de setembre de 2010.